8.2 **Surface Smoothing**

Hao Li

http://cs621.hao-li.com

Mesh Optimization

Smoothing

- Low geometric noise

Fairing

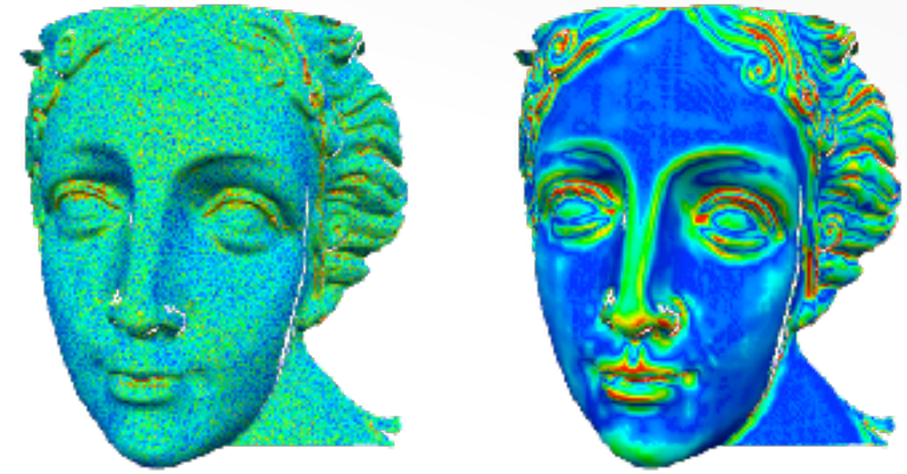
- Simplest shape

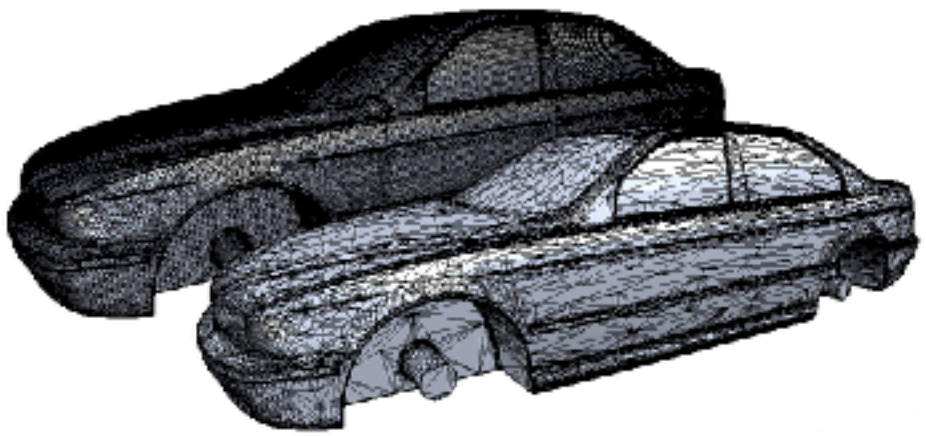
Decimation

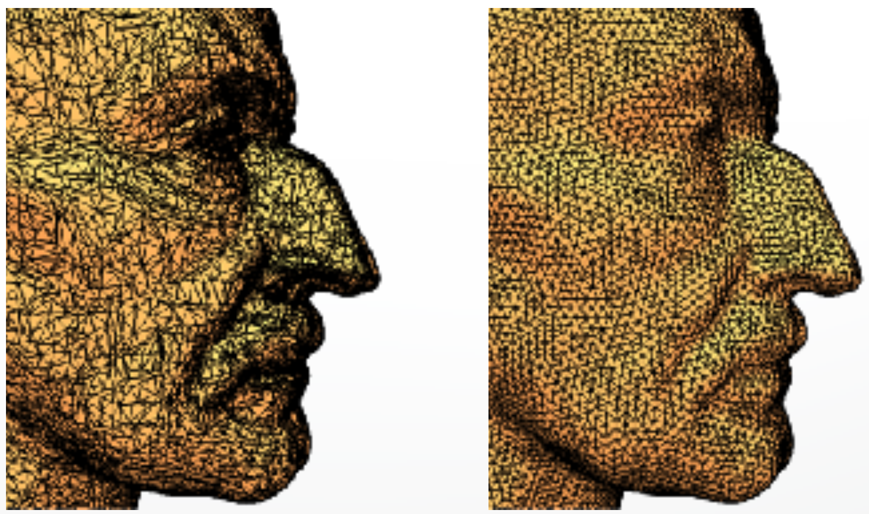
- Low complexity

Remeshing

- Triangle Shape

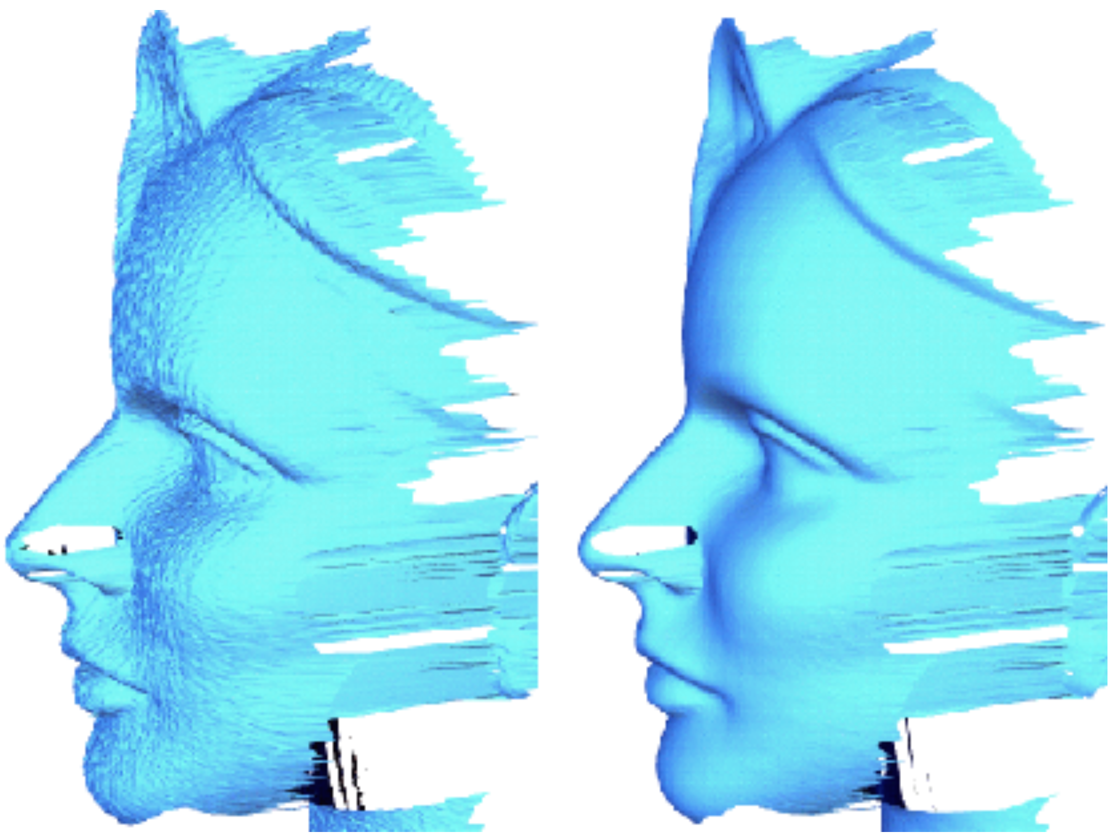






Mesh Smoothing

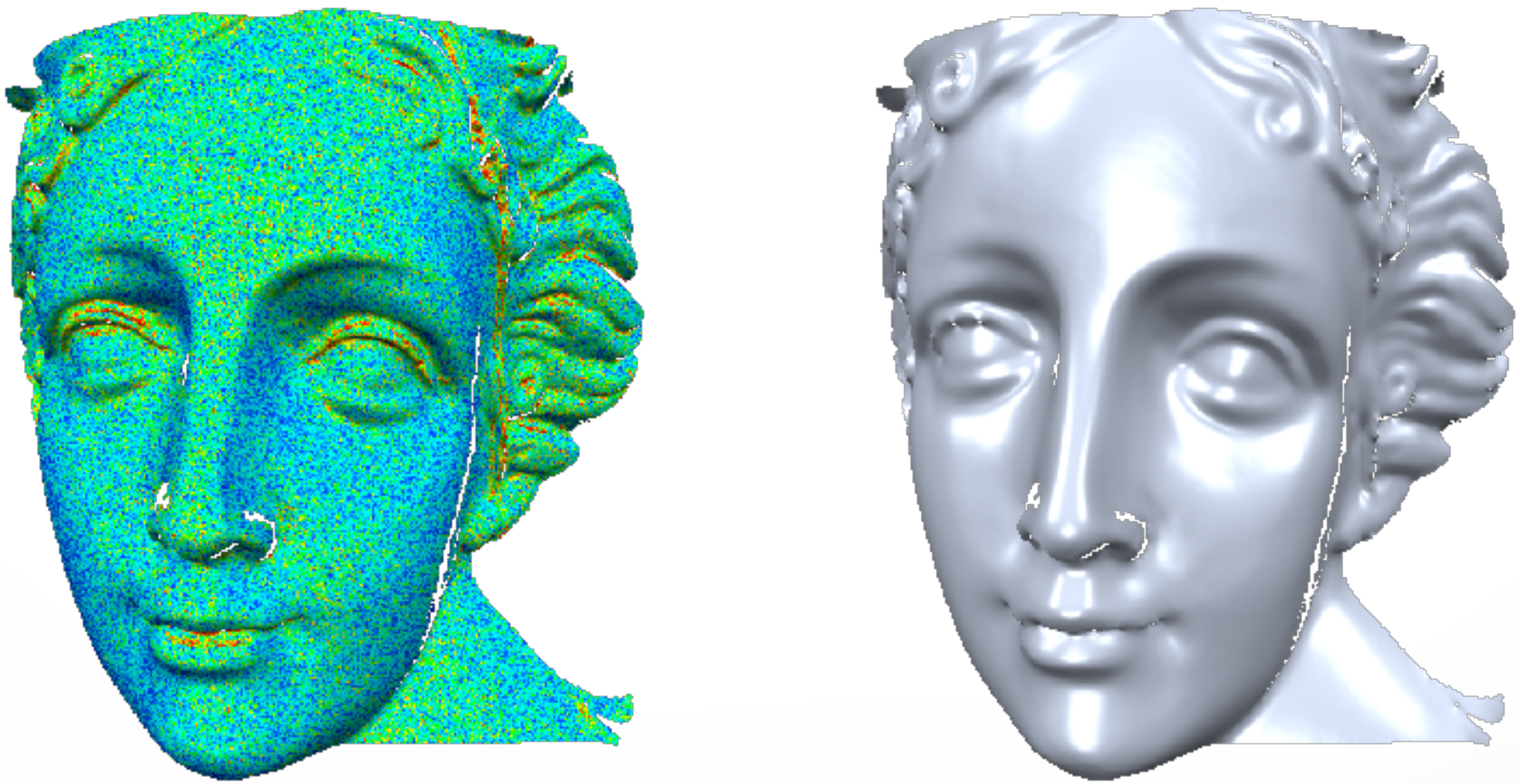
Filter out high frequency noise



Desbrun, Meyer, Schroeder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 99

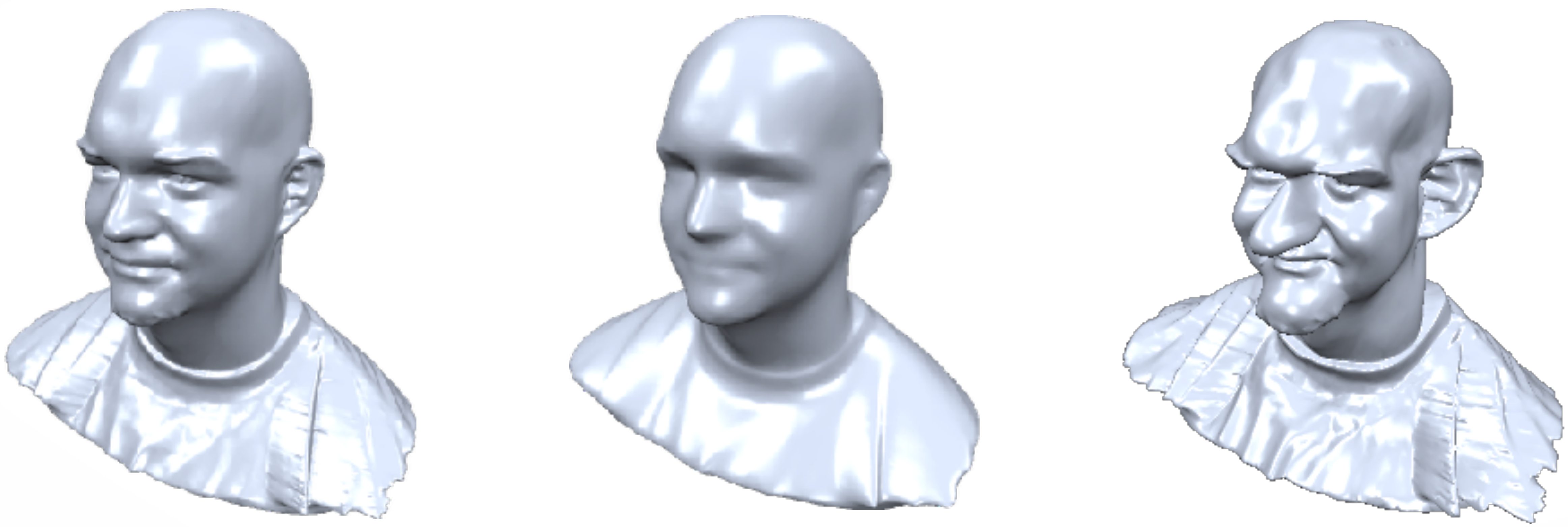
Mesh Smoothing

Filter out high frequency noise



Mesh Smoothing

Advanced Filtering

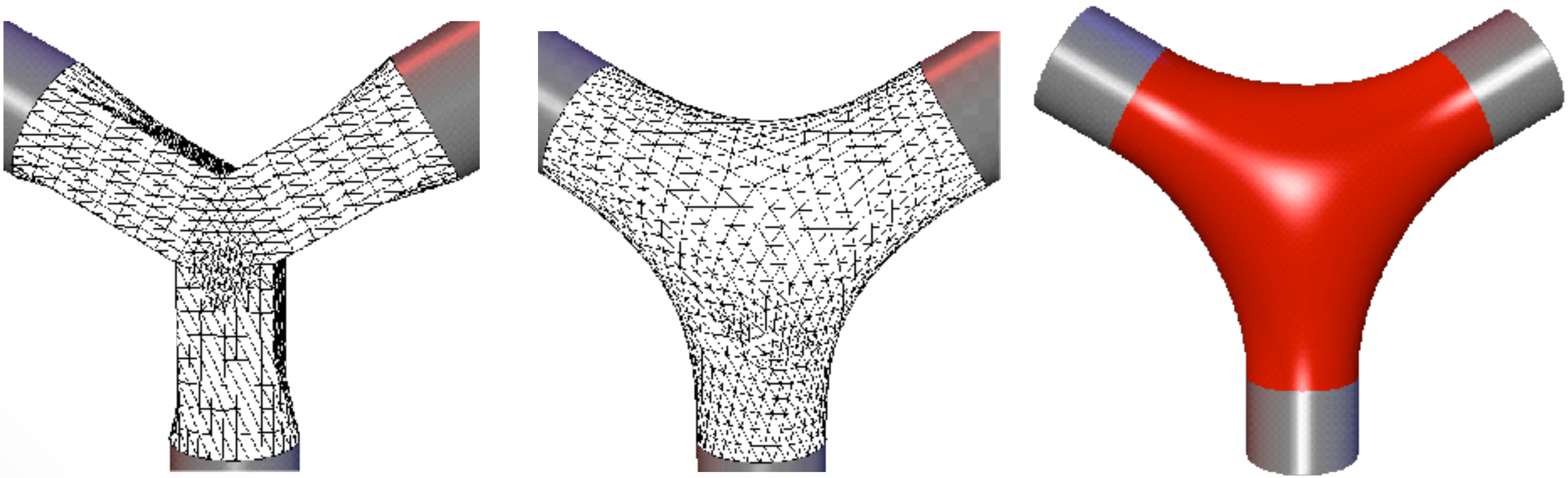


input data　　low pass　　exaggerate

Kim, Rosignac: *Geofilter: Geometric Selection of Mesh Filter Parameters*, Eurographics 05

Mesh Smoothing

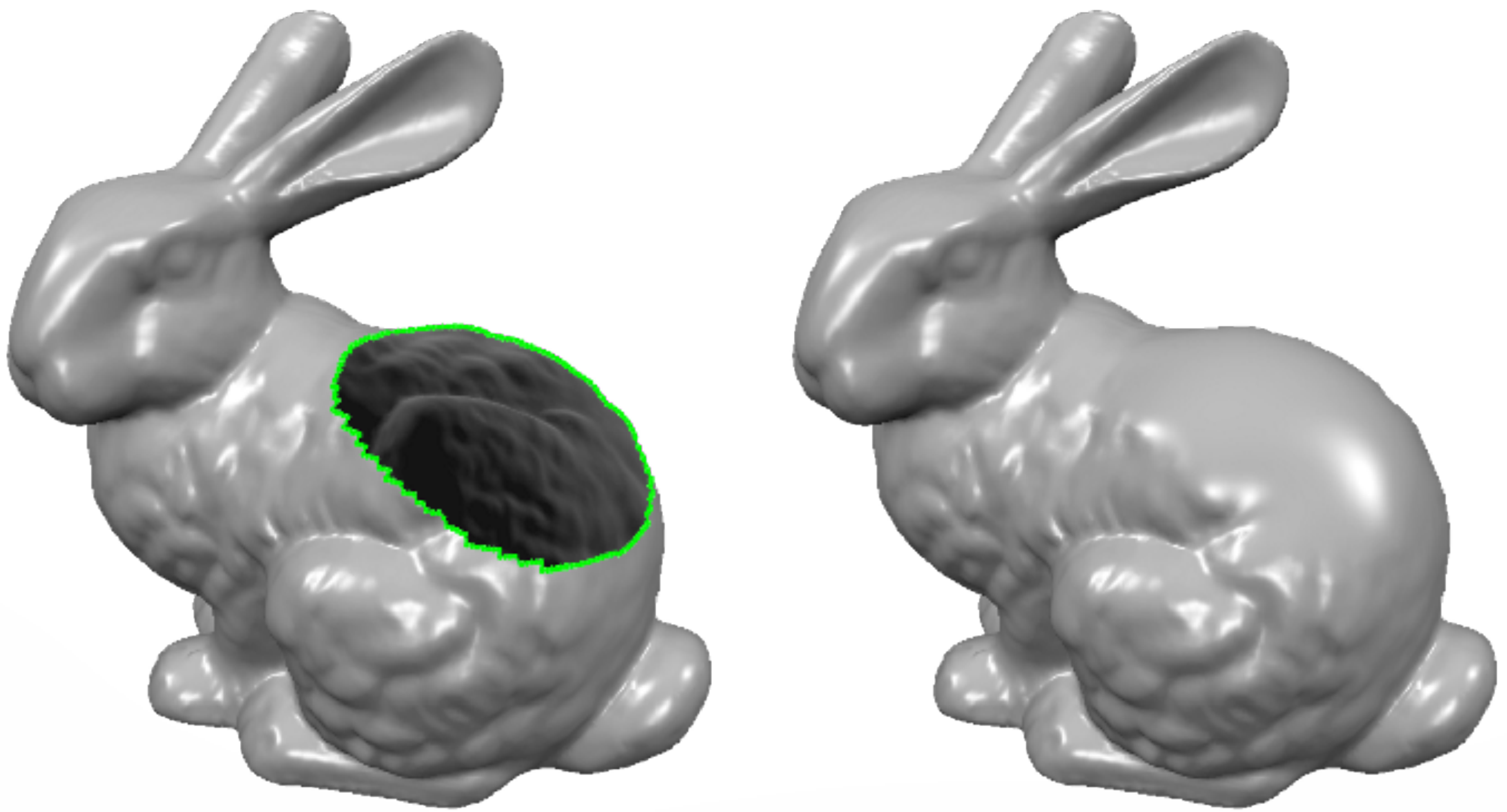
Fair Surface Design



Schneider, Kobbelt: *Geometric fairing of irregular meshes for free-form surface design*, CAGD 18(4), 2001

Mesh Smoothing

Hole filling with energy-minimizing patches

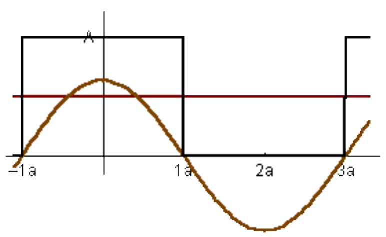


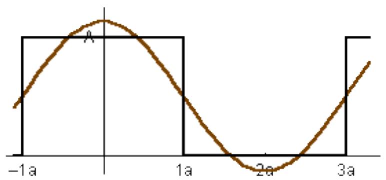
Outline

- **Spectral Analysis**
- **Diffusion Flow**
- **Energy Minimization**

Fourier Transform

Represent a function as a weighted sum of sines and cosines



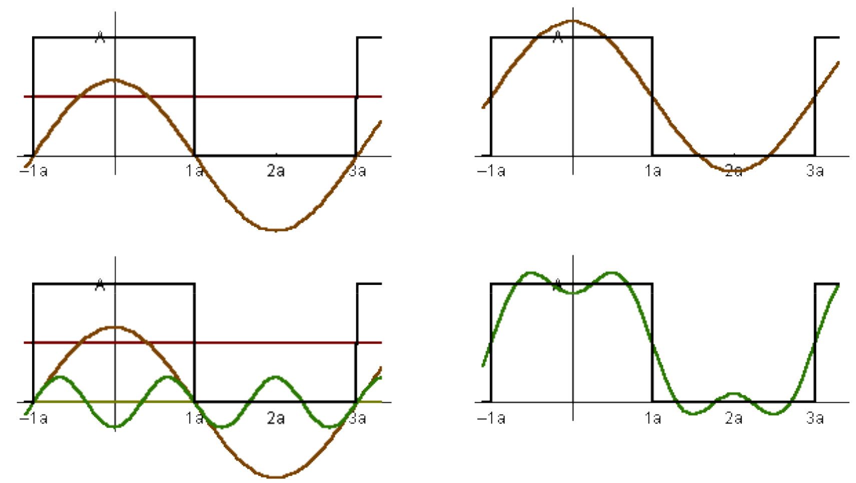


Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x)$$

Fourier Transform

Represent a function as a weighted sum of sines and cosines



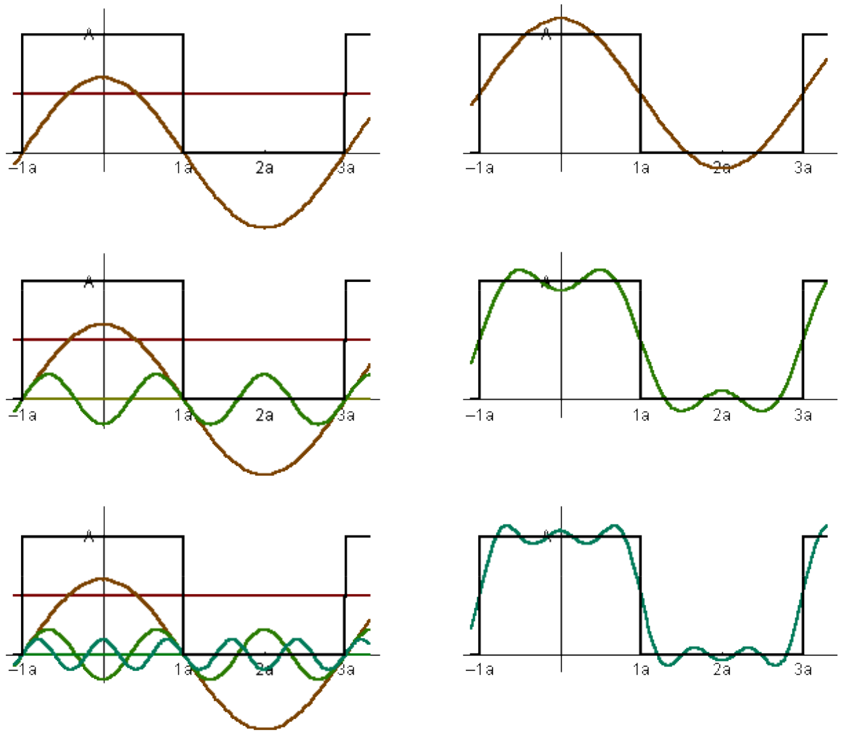
Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x)$$

Fourier Transform

Represent a function as a weighted sum of sines and cosines

Joseph Fourier 1768 - 1830

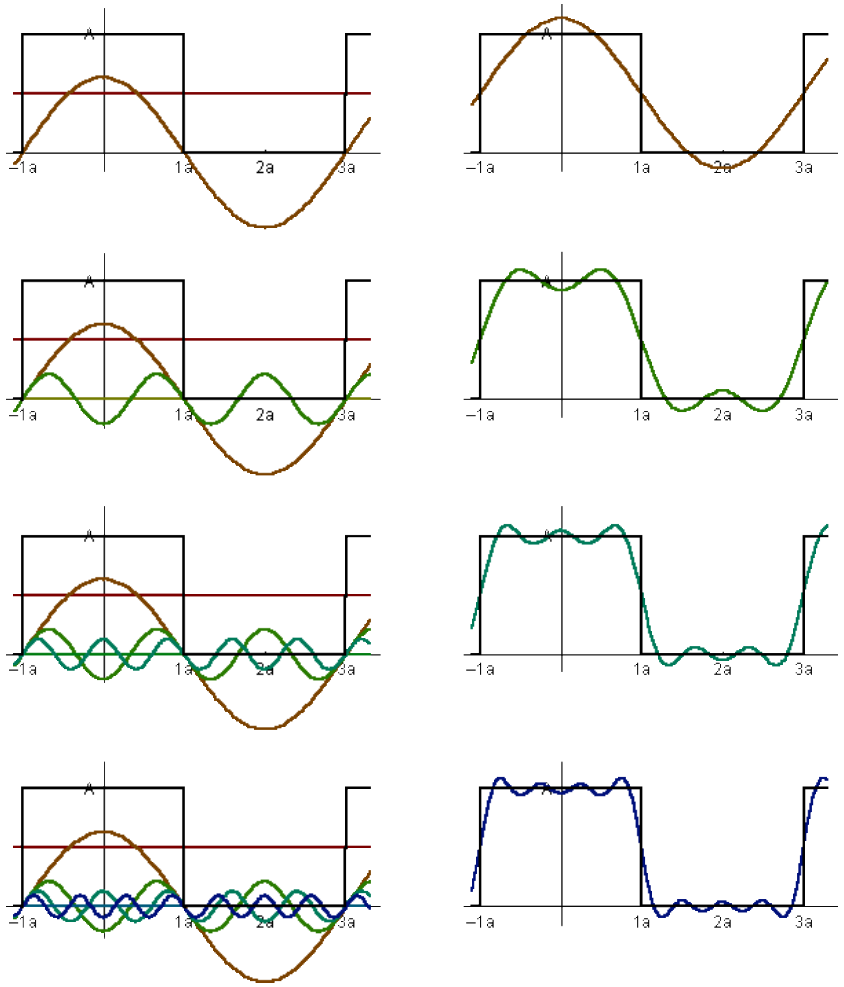


$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x)$$

Fourier Transform

Represent a function as a weighted sum of sines and cosines

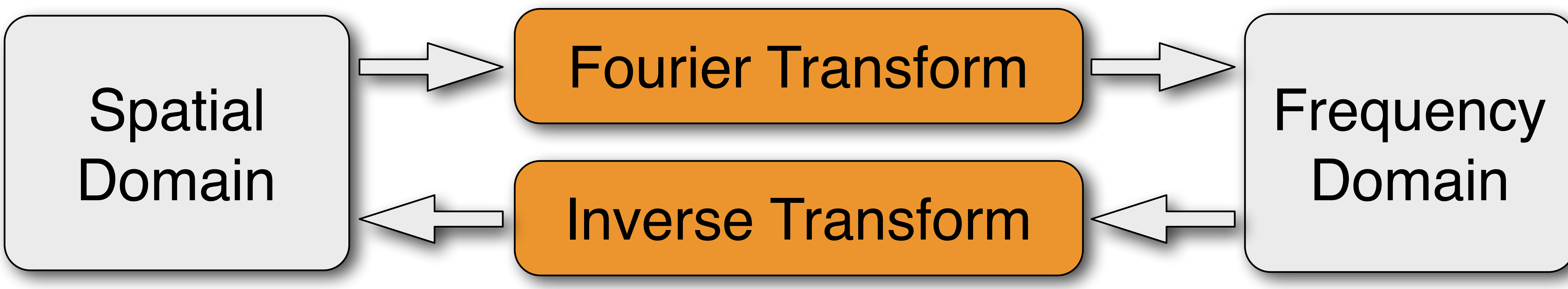
Joseph Fourier 1768 - 1830



$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \ldots$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$



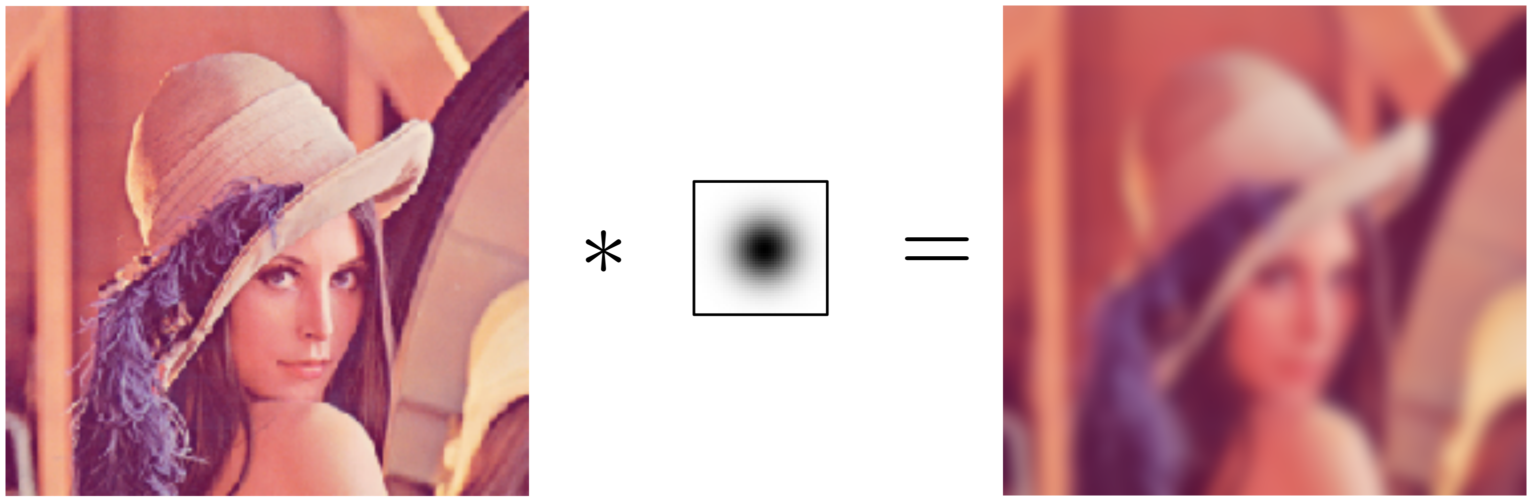
$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{2\pi \mathrm{i} \omega x}\, \mathrm{d}\omega$$

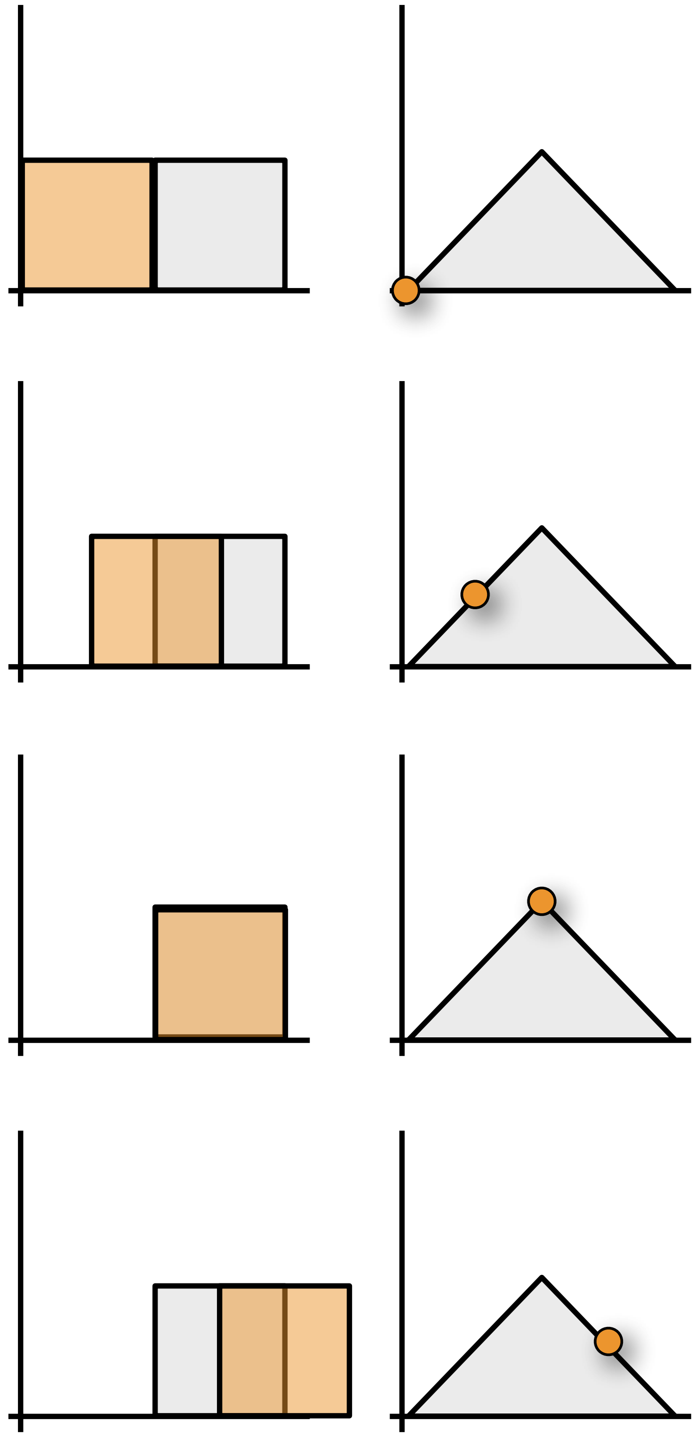
Convolution

Smooth signal by convolution with a kernel function

$$h(x) \;=\; f * g \;:=\; \int f(y) \cdot g(x-y)\, \mathrm{d}y$$

Example: Gaussian blurring





Convolution

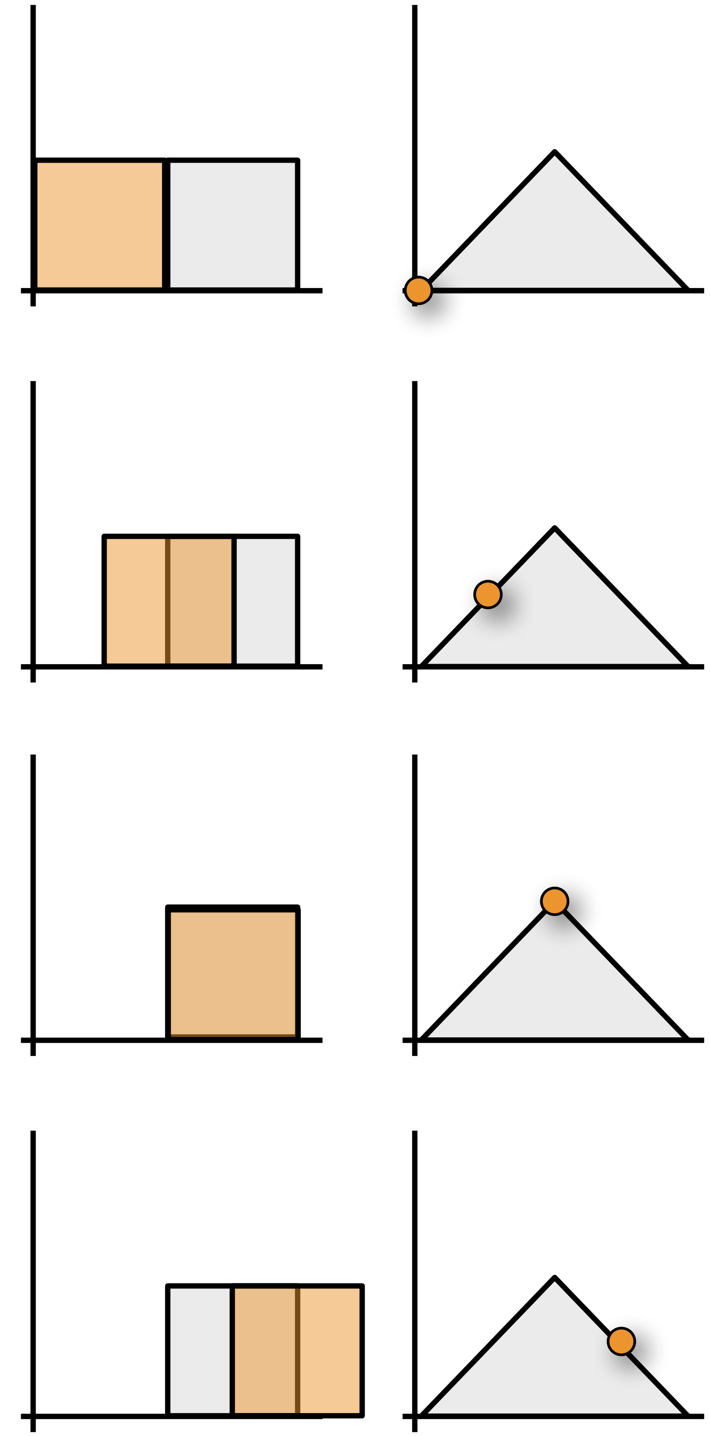
Smooth signal by convolution with a kernel function

$$h(x) \;=\; f * g \;:=\; \int f(y) \cdot g(x-y)\,\mathrm{d}y$$

Convolution in spatial domain ⇔

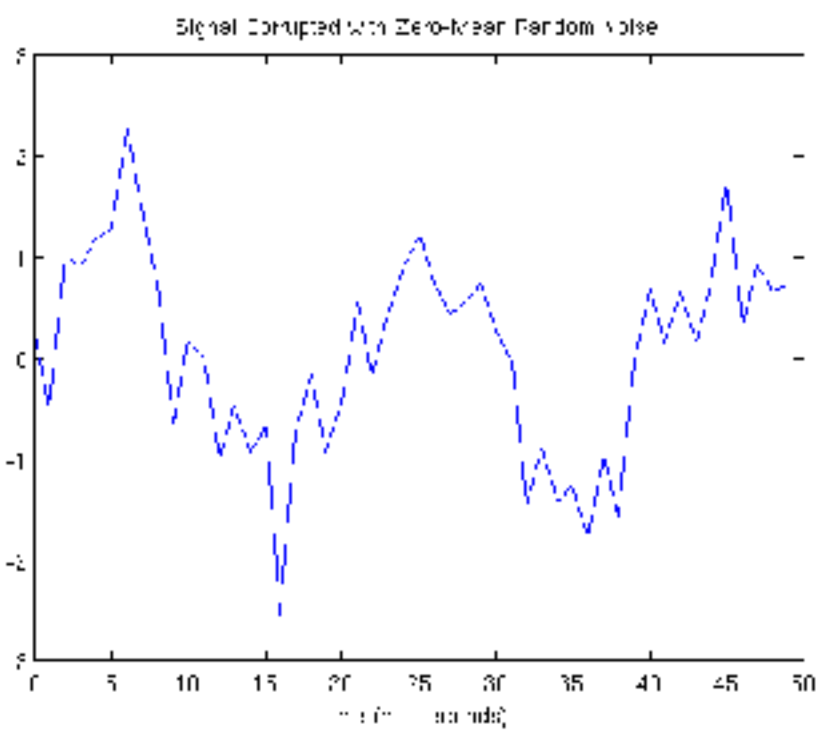
Multiplication in frequency domain

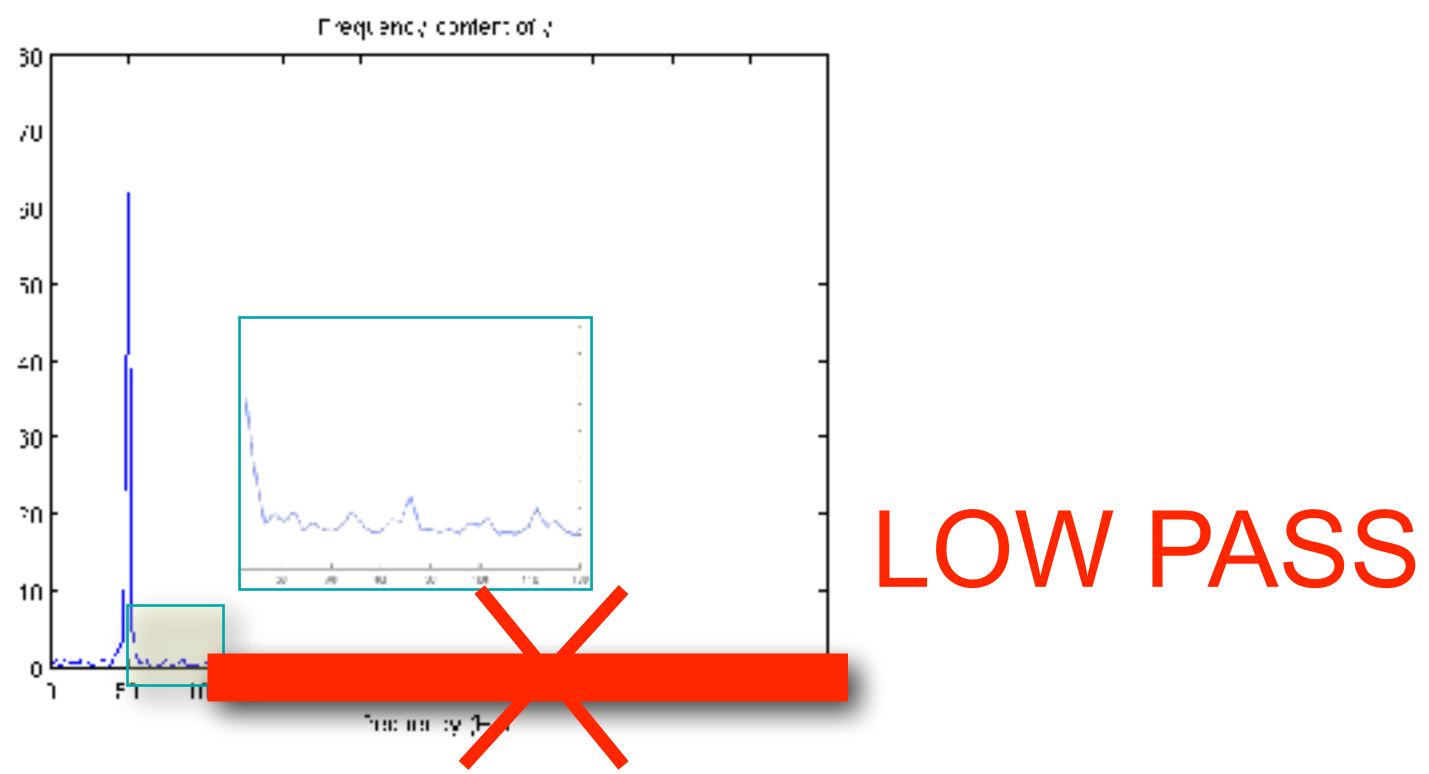
$$H(\omega) \;=\; F(\omega) \cdot G(\omega)$$



Fourier Analysis

Low-pass filter discards high frequencies





spatial domain

frequency domain

Fourier Transform

Spatial domain $f(x)$ → **Frequency domain** $F(w)$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$

Multiply by low-pass filter $G(w)$

$$F(\omega) \; \leftarrow \; F(\omega) \cdot G(\omega)$$

Frequency domain $F(w)$ → **Spatial domain** $f(x)$

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{2\pi \mathrm{i} \omega x} \; \mathrm{d}\omega$$

Fourier Transform

Consider L^2-function space with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, \mathrm{d}x$$

Complex "waves" build an orthonormal basis

$$e_\omega(x) \; := \; \mathrm{e}^{-2\pi \mathrm{i} \omega x} \; = \; \cos(2\pi\omega x) - \mathrm{i}\sin(2\pi\omega x)$$

Fourier transform is a change of basis

$$f(x) = \sum_{\omega=-\infty}^{\infty} \langle f, e_\omega \rangle \; e_\omega \; \mathrm{d}\omega \quad \Rightarrow \quad f(x) = \int_{-\infty}^{\infty} \langle f, e_\omega \rangle \; e_\omega \; \mathrm{d}\omega$$

Fourier Analysis on Meshes?

- Only applicable to parametric patches
- Generalize frequency to the discrete setting
- Complex waves are Eigenfunctions of Laplace

$$\Delta\left(e^{2\pi \mathrm{i}\omega x}\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\, e^{2\pi \mathrm{i}\omega x} = -\left(2\pi\omega\right)^2 e^{2\pi \mathrm{i}\omega x}$$

Use Eigenfunctions of discrete Laplace-Beltrami

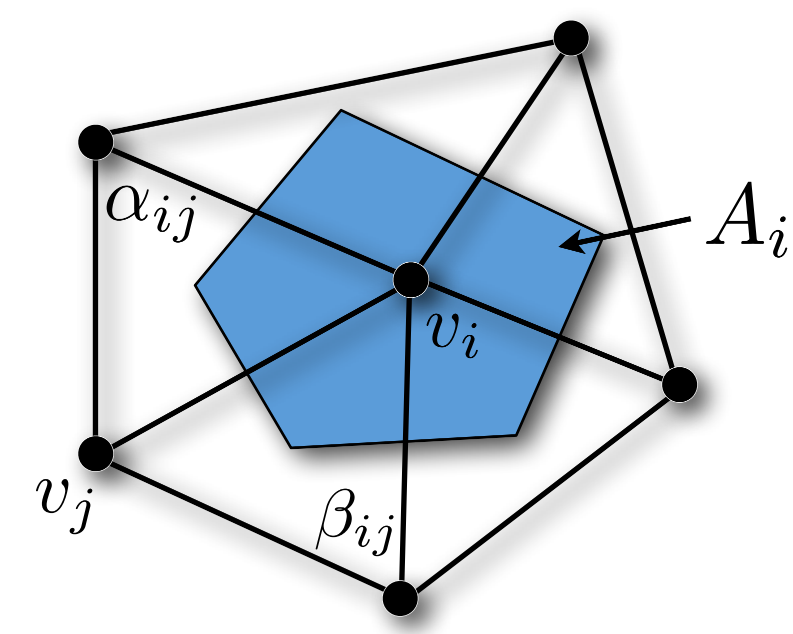
Discrete Laplace-Beltrami

- Function values sampled at mesh vertices

$$\mathbf{f} \;=\; [f_1, f_2, \ldots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

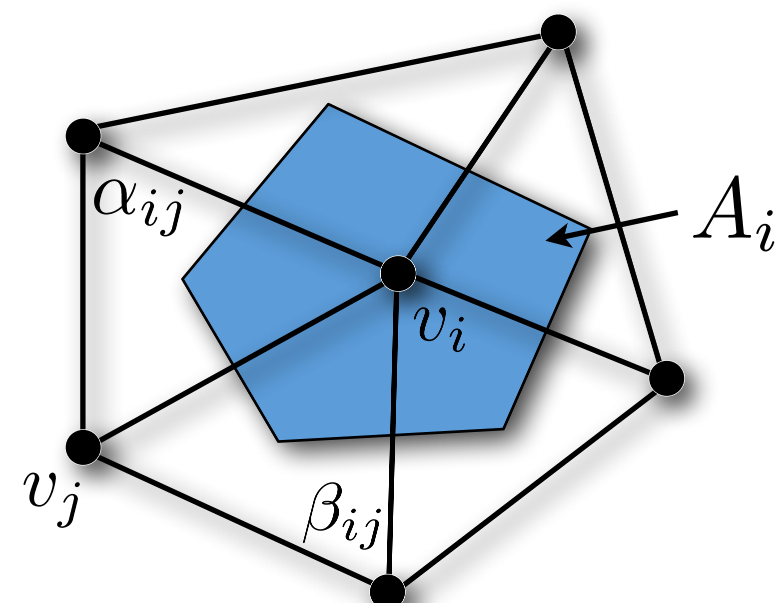
$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) \, (f(v_j) - f(v_i))$$



Discrete Laplace-Beltrami

- Discrete Laplace Operator (per mesh)
  - Sparse matrix $\mathbf{L} = \mathbf{D}\mathbf{M} \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} \vdots \\ \Delta_{\mathcal{S}} f(v_i) \\ \vdots \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} \vdots \\ f(v_i) \\ \vdots \end{pmatrix}$$

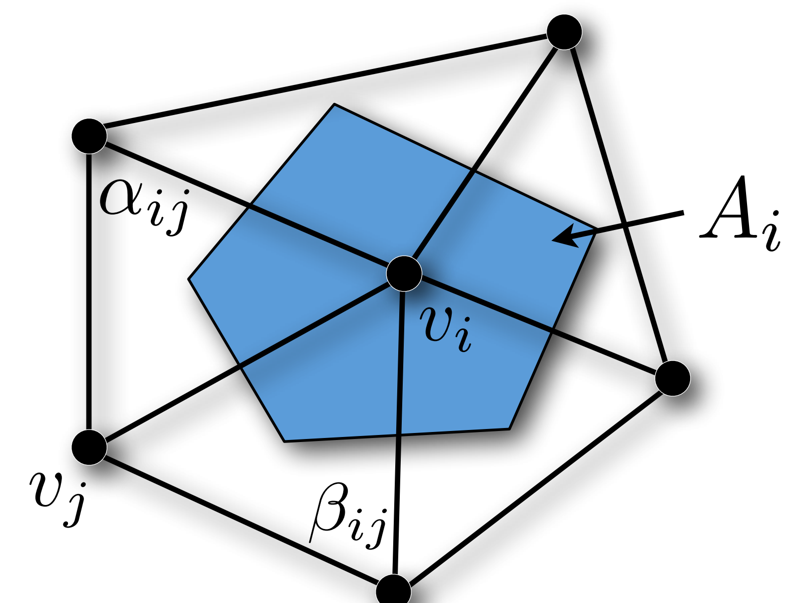


Discrete Laplace-Beltrami

- Discrete Laplace Operator (per mesh)
  - Sparse matrix $\mathbf{L} = \mathbf{DM} \in \mathbb{R}^{n \times n}$

$$\mathbf{M}_{ij} = \begin{cases} \cot\alpha_{ij} + \cot\beta_{ij}, & i \neq j\ ,\ j \in \mathcal{N}_1(v_i) \\ -\sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{D} = \operatorname{diag}\left(\ldots, \frac{1}{2A_i}, \ldots\right)$$



Discrete Laplace-Beltrami

- Function values sampled at mesh vertices

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- Discrete Laplace-Beltrami (per vertex)

$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij})\,(f(v_j) - f(v_i))$$

- Discrete Laplace-Beltrami matrix $\mathbf{L} = \mathbf{D}\mathbf{M} \in \mathbb{R}^{n \times n}$
  - Eigenvectors are **natural vibrations**
  - Eigenvalues are **natural frequencies**

Discrete Laplace-Beltrami

- Discrete Laplace-Beltrami matrix $\mathbf{L} = \mathbf{DM} \in \mathbb{R}^{n \times n}$
  - Eigenvectors are **natural vibrations**
  - Eigenvalues are **natural frequencies**

Spectral Analysis

- Setup Laplace-Beltrami matrix $\mathbf{L}$
- Compute k smallest eigenvectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$
- Reconstruct mesh from those (component-wise)

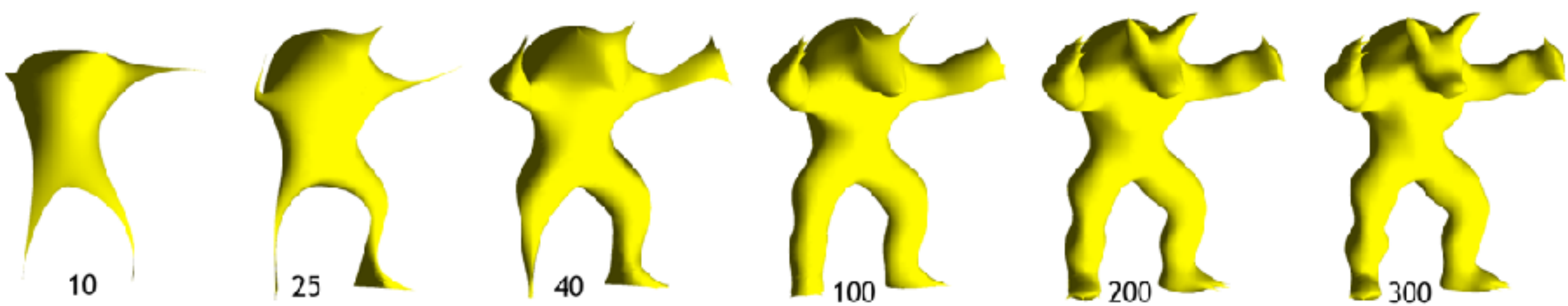
$$\mathbf{x} := [x_1, \ldots, x_n] \qquad \mathbf{y} := [y_1, \ldots, y_n] \qquad \mathbf{z} := [z_1, \ldots, z_n]$$

$$\mathbf{x} \leftarrow \sum_{i=1}^{k} \left(\mathbf{x}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \mathbf{y} \leftarrow \sum_{i=1}^{k} \left(\mathbf{y}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \mathbf{z} \leftarrow \sum_{i=1}^{k} \left(\mathbf{z}^T \mathbf{e}_i\right) \mathbf{e}_i$$

Spectral Analysis

- Setup Laplace-Beltrami matrix $\mathbf{L}$
- Compute k smallest eigenvectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$
- Reconstruct mesh from those (component-wise)

Too complex for large meshes!



Bruno Levy: *Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry*, Shape Modeling and Applications, 2006

Outline

- Spectral Analysis
- **Diffusion Flow**
- Energy Minimization

Diffusion Flow on Height Fields

Diffusion equation (this one is heat equation)

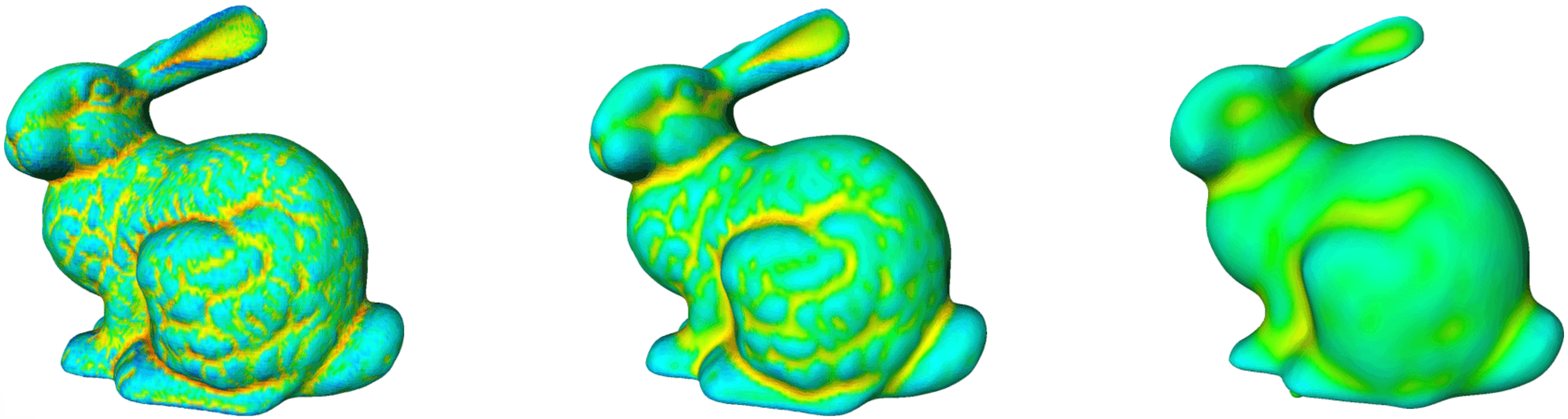
diffusion constant

$$\frac{\partial f}{\partial t} = \lambda \Delta f$$

Laplace operator

Diffusion Flow on Meshes

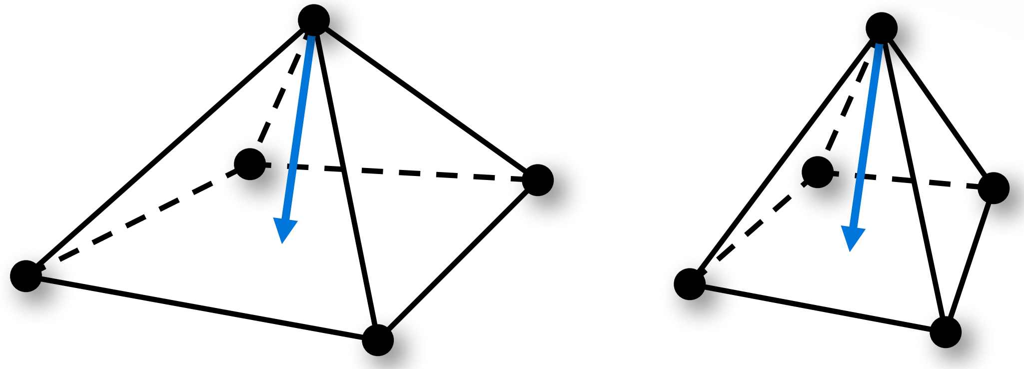
Iterate $\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$

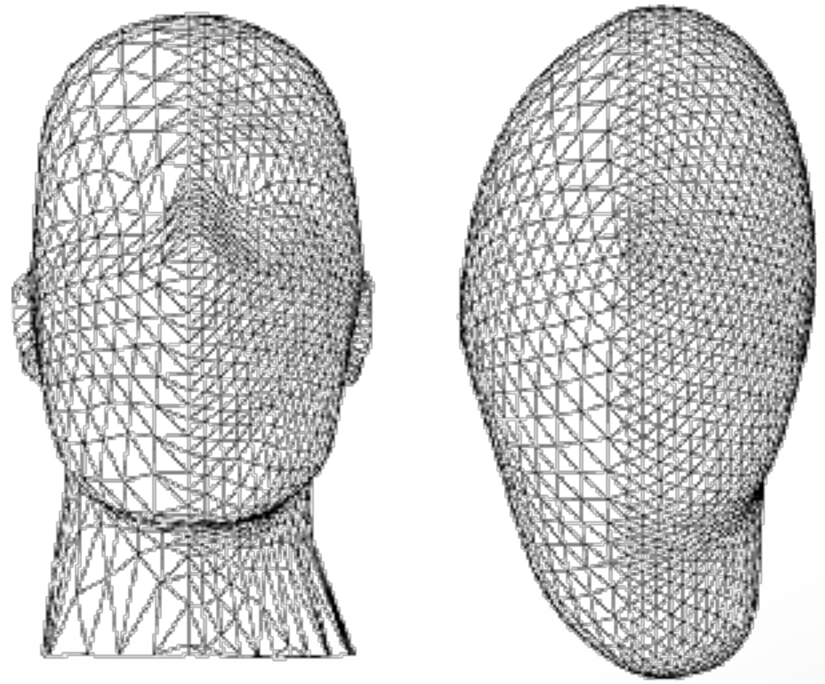


0 Iterations 5 Iterations 20 Iterations

Uniform Laplace Discretization

- Smoothes geometry and triangulation
- Can be non-zero even for planar triangulation
- Vertex drift can lead to distortions
- Might be desired for mesh regularization





Desbrun et al., Siggraph 1999

Mean **Curvature Flow**

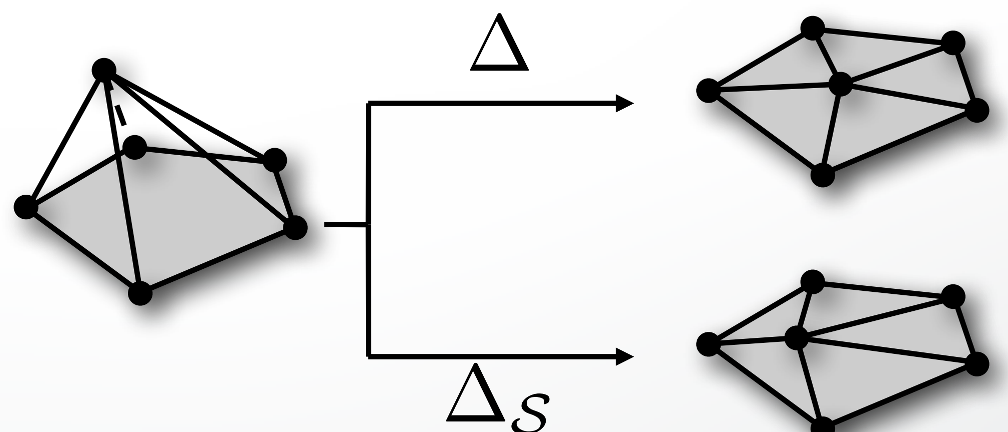
- Use diffusion flow with Laplace-Beltrami

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta_{\mathcal{S}} \mathbf{p}$$

- Laplace-Beltrami is parallel to surface normal

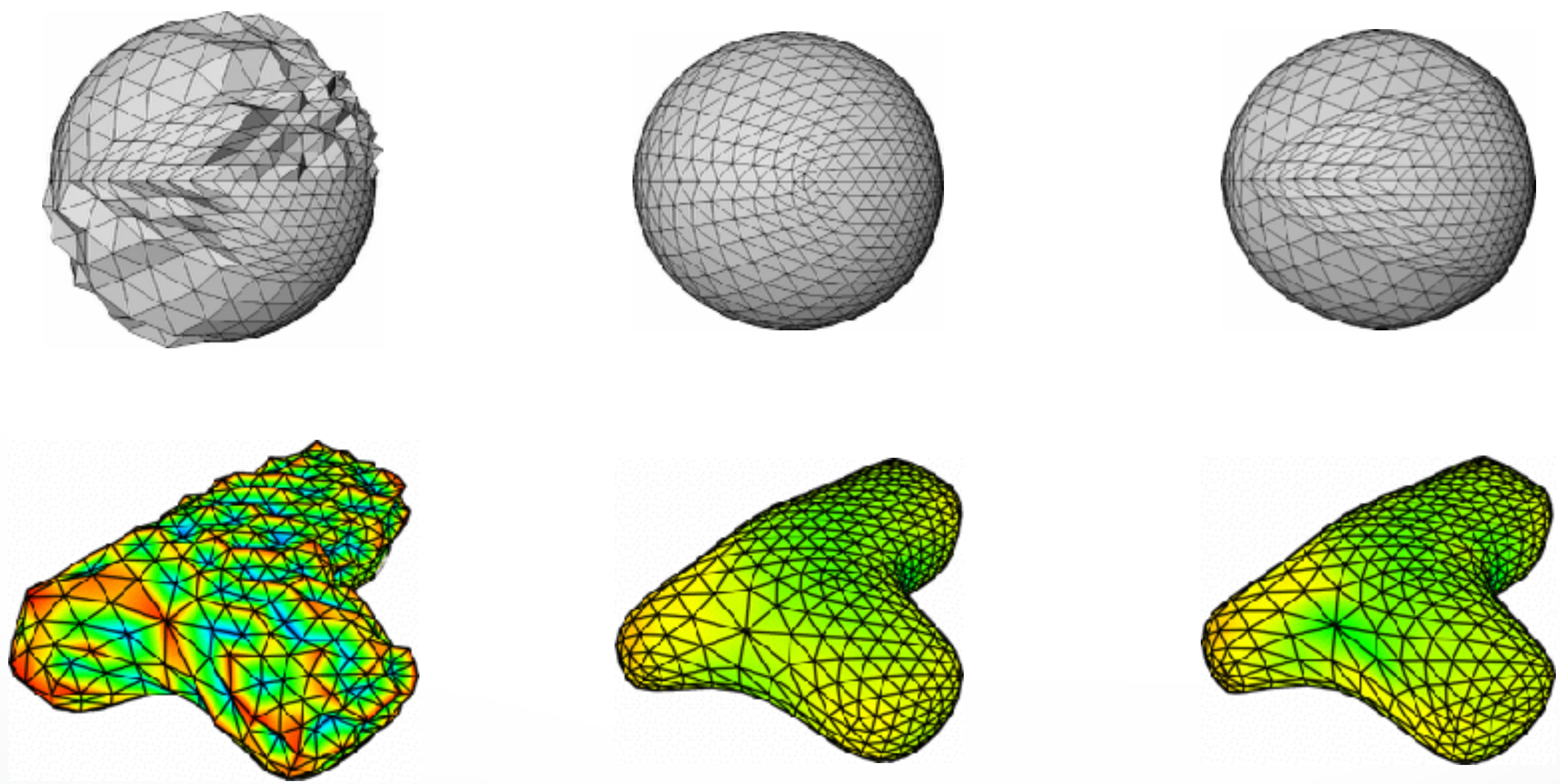
$$\frac{\partial \mathbf{p}}{\partial t} = -2\lambda H \mathbf{n}$$

Avoids vertex drift on surface



Mean **Curvature Flow**

input data | uniform Laplace | Laplace-Beltrami



Numerical **Integration**

- Write update $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$ in matrix notation

$$\mathbf{P}^{(t)} = \left(\mathbf{p}_1^{(t)}, \ldots, \mathbf{p}_n^{(t)}\right)^T \in \mathbb{R}^{n \times 3}$$

- Corresponds to explicit integration

$$\mathbf{P}^{(t+1)} \;=\; (\mathbf{I} + \lambda \mathbf{L})\,\mathbf{P}^{(t)}$$

Requires small λ for stability!

- Implicit integration is unconditionally stable

$$(\mathbf{I} - \lambda \mathbf{L})\,\mathbf{P}^{(t+1)} \;=\; \mathbf{P}^{(t)}$$

(backward Euler method)

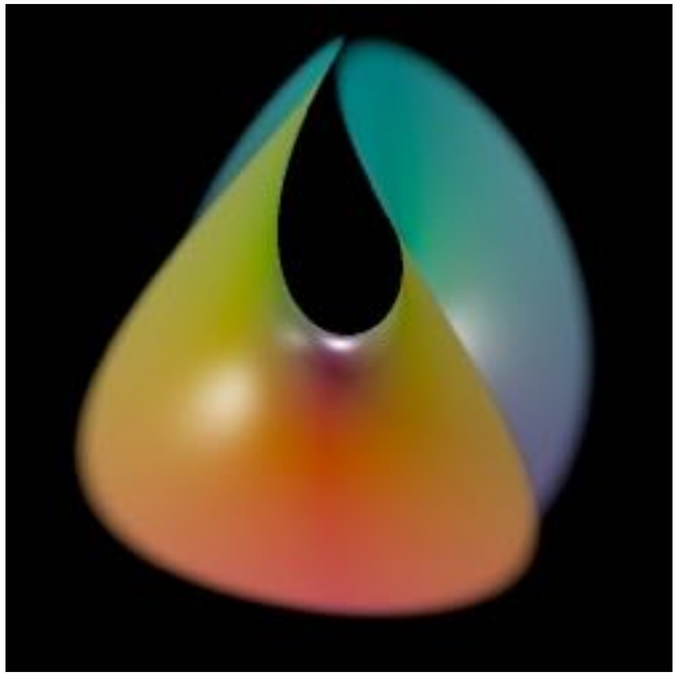
Outline

- Spectral Analysis
- Diffusion Flow
- **Energy Minimization**

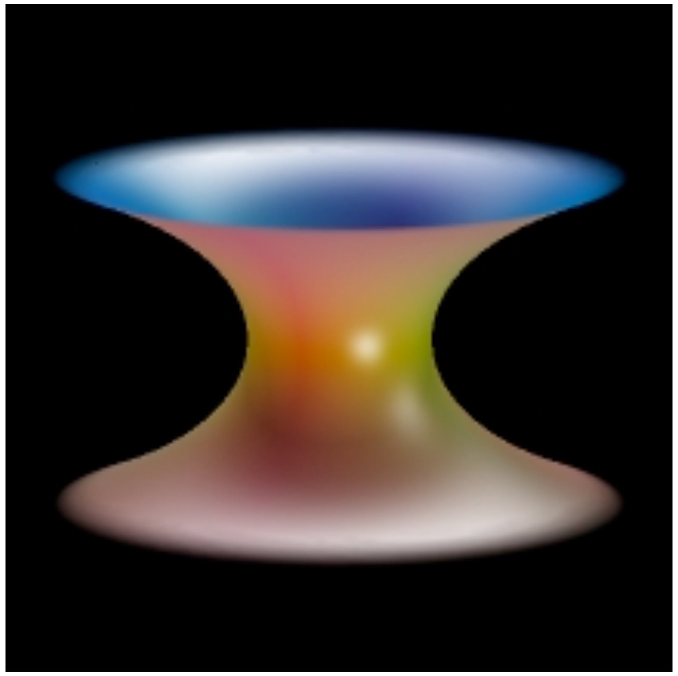
Fairness

- Idea: Penalize “unaesthetic behavior”
- Measure fairness
  - Principle of the simplest shape
  - Physical interpretation
- Minimize some fairness functional
  - Surface area, curvature
  - Membrane energy, thin plate energy

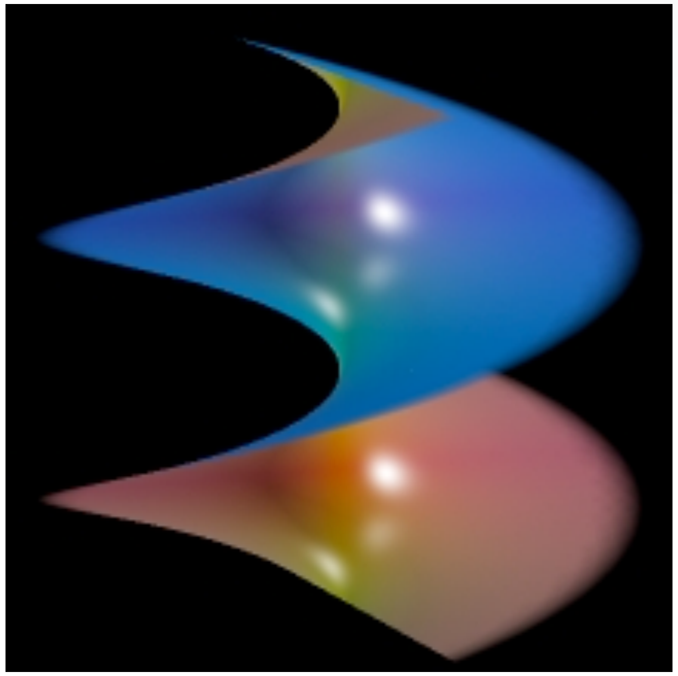
Minimal Surfaces



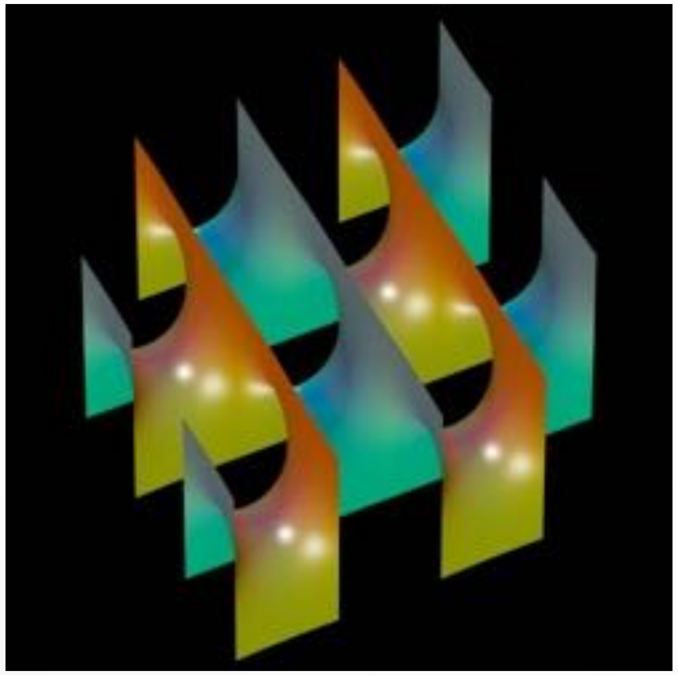
Enneper’s Surface



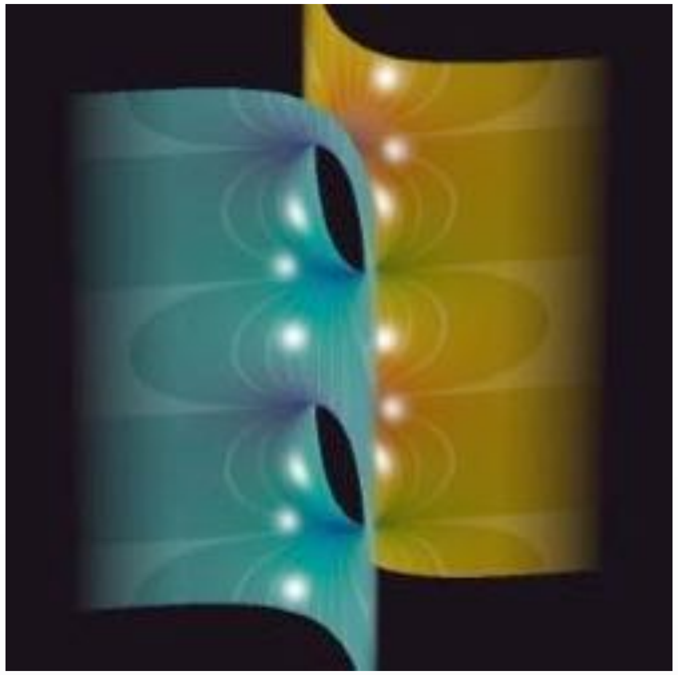
Catenoid



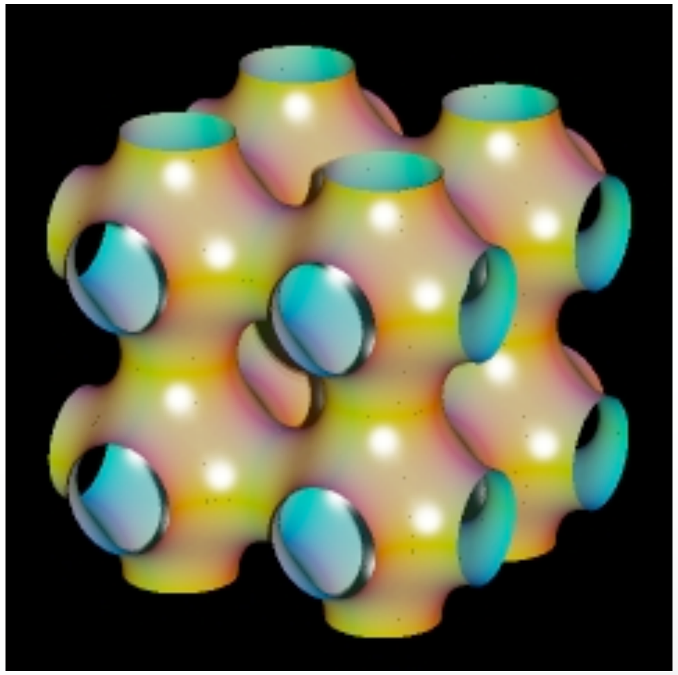
Helicoid



Scherk’s First Surface



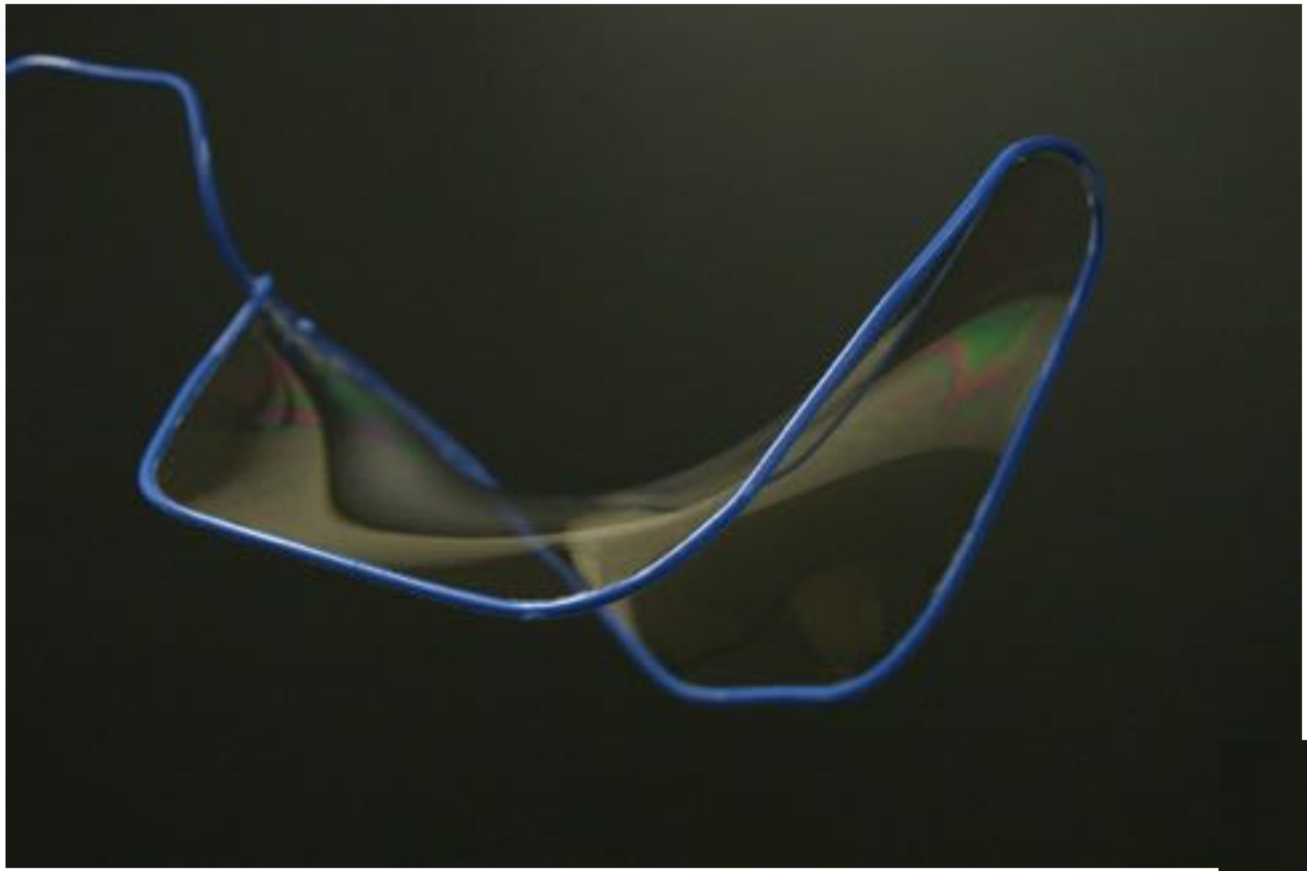
Scherk’s Second Surface

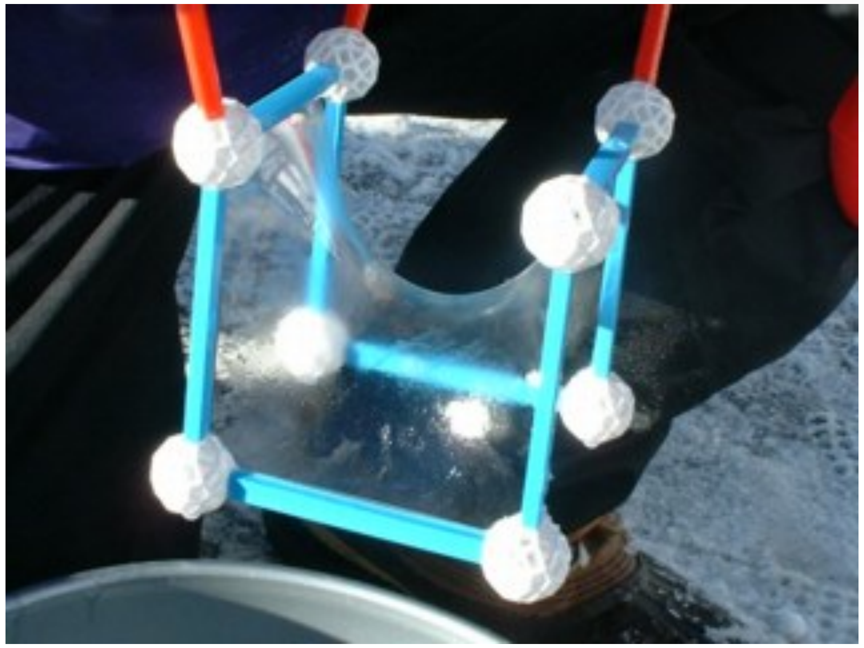


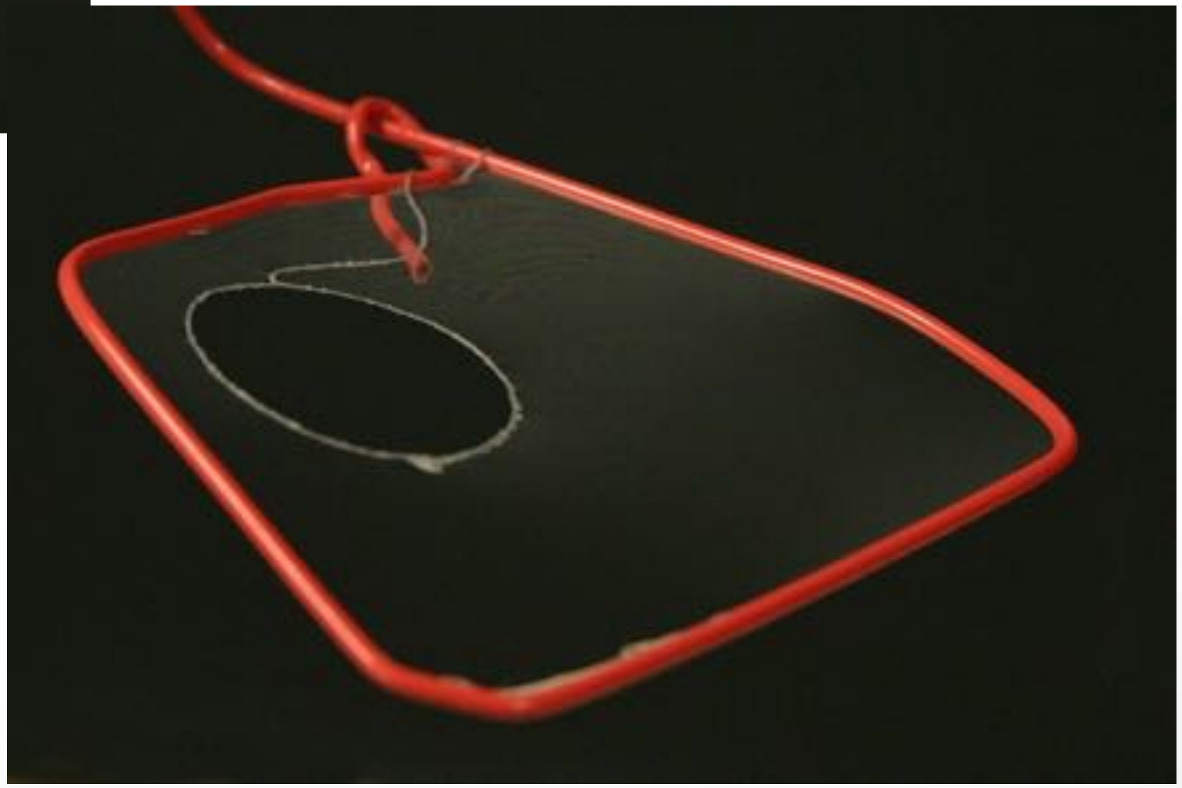
Schwarz P Surface

source: http://www.msri.org/about/sgp/jim/geom/minimal/library/index.html

Soap Films







Surface Area

$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - \left(\mathbf{x}_u^T \mathbf{x}_v\right)^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \sqrt{EG - F^2} \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

infinitesimal
Area

cross product → determinant with unit vectors → area

Non-Linear Energies

- Membrane energy (surface area)

$$\int_{\mathcal{S}} \mathrm{d}A \;\to\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}$$

- Thin-plate surface (curvature)

$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \,\mathrm{d}A \;\to\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}, \;\; \mathbf{n}(\delta\mathcal{S}) = \mathbf{d}$$

- Too complex… simplify energies

Membrane Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$$

- Membrane energy (surface Area)

$$\int_\Omega \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, \mathrm{d}u\mathrm{d}v \;\to\; \min$$

Variational Calculus in 1D

- 1D membrane energy

$$L(f) = \int_a^b {f'}^2(x)\ \mathrm{d}x \ \rightarrow \ \min$$

- Add test function u with $u(a) = u(b) = 0$

$$L(f + \lambda u) = \int_a^b (f' + \lambda u')_f^2 = \int_a^b {f'}^2 + 2\lambda f' u' + \lambda^2 {u'}^2$$

- If f minimizes L , the following has to vanish

$$\left.\frac{\partial L(f + \lambda u)}{\partial \lambda}\right|_{\lambda=0} = \int_a^b 2 f' u' \ \stackrel{!}{=}\ 0$$

Variational Calculus in 1D

- Has to vanish for any u with $u(a) = u(b) = 0$

$$\int_a^b f'u' \;=\; \underbrace{[f'u]_a^b}_{=0} - \int_a^b f''u \;\stackrel{!}{=}\; 0 \quad \forall u$$

$$\boxed{\int_0^1 f'g = [fg]_0^1 - \int_0^1 fg'}$$

- Only possible if

$$f'' \;=\; \Delta f \;=\; 0$$

Euler-Lagrange equation

Bivariate Variational Caculus

- Find minimum of functional

$$\operatorname*{argmin}_{f} \int_{\Omega} L\left(f_{uu}, f_{vv}, f_{u}, f_{v}, f, u, v\right)$$

- Euler-lagrange PDE defines the minimizer

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial u}\frac{\partial L}{\partial f_u} - \frac{\partial}{\partial v}\frac{\partial L}{\partial f_v} + \frac{\partial^2}{\partial u^2}\frac{\partial L}{\partial f_{uu}} + \frac{\partial^2}{\partial u \partial v}\frac{\partial L}{\partial f_{uv}} + \frac{\partial^2}{\partial v^2}\frac{\partial L}{\partial f_{vv}} = 0$$

Again, subject to suitable boundary constraints

Bivariate Variational Caculus

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\int_\Omega \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, \mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

- Variational caculus

$$\Delta \mathbf{p} = 0$$

Thin-Plate Surface

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$$

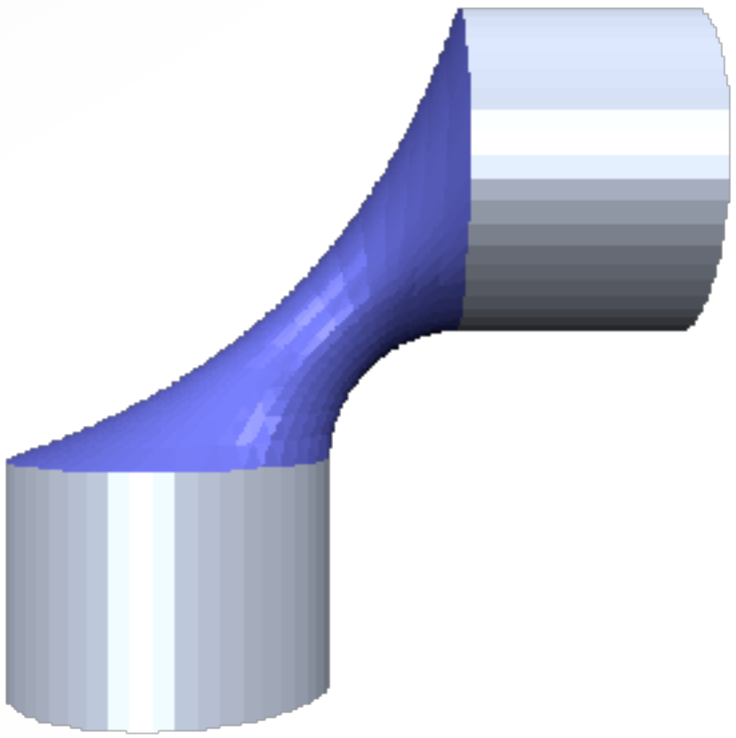
- Thin-plate energy (curvature)

$$\int_{\Omega} \|\mathbf{p}_{uu}\|^2 + 2\,\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \,\mathrm{d}u\mathrm{d}v \;\to\; \min$$

- Variational caculus

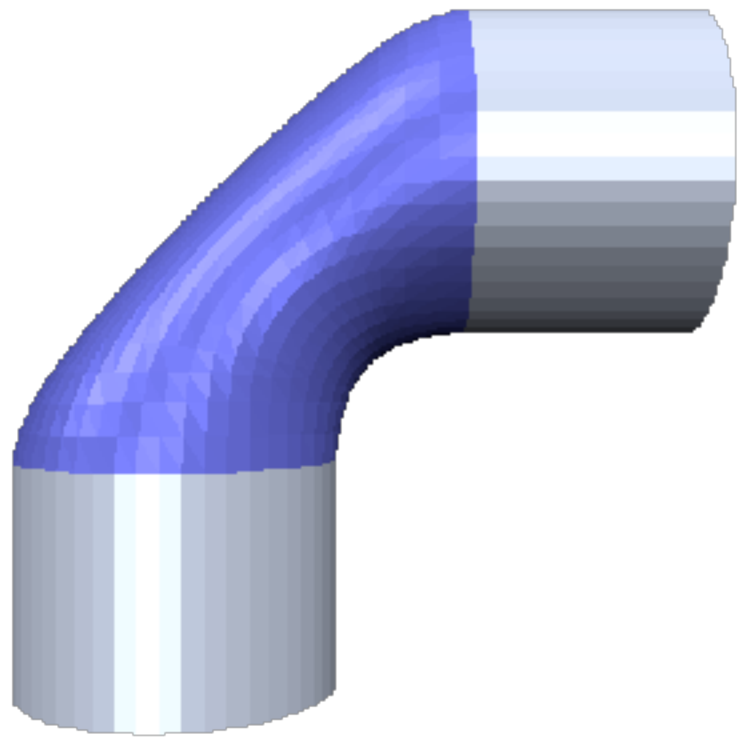
$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals



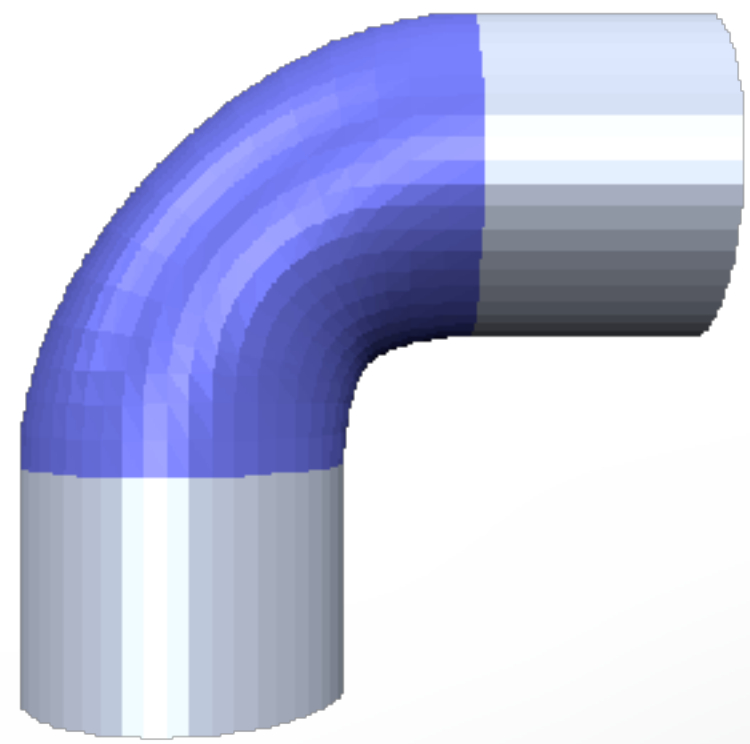
Membrane

$$\Delta_{\mathcal{S}} \mathbf{p} = 0$$



Thin Plate

$$\Delta_{\mathcal{S}}^2 \mathbf{p} = 0$$



$$\Delta_{\mathcal{S}}^3 \mathbf{p} = 0$$

Analysis

- Minimizer surfaces satisfy Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}^{k}\mathbf{p} = 0$$

- They are stationary surfaces of Laplacian flow

$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_{\mathcal{S}}^{k}\mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

Literature

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- Levy: Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry, Shape Modeling and Applications, 2006
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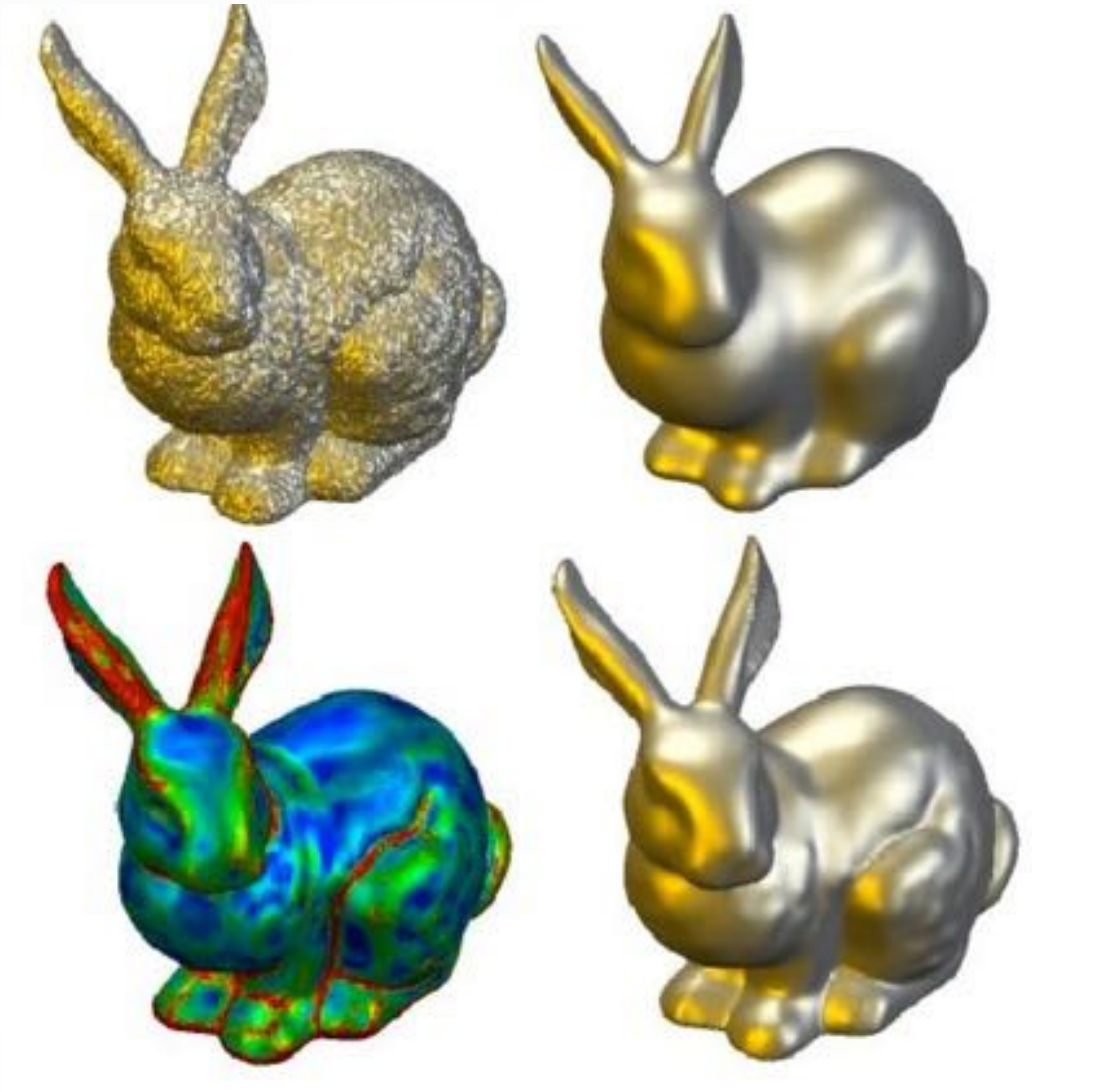
Advanced Methods



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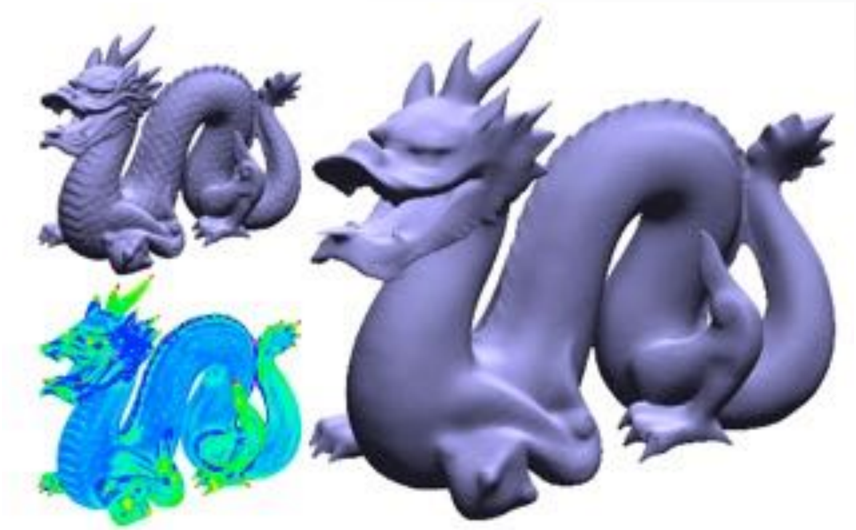

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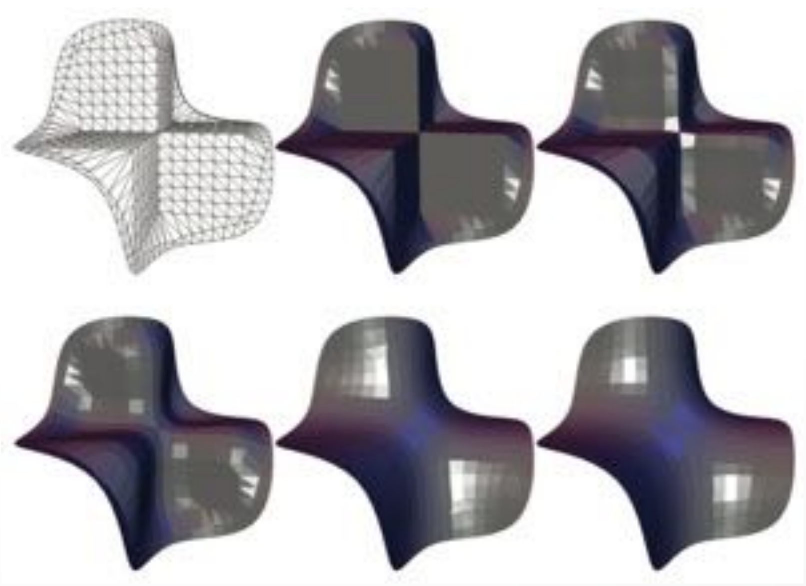

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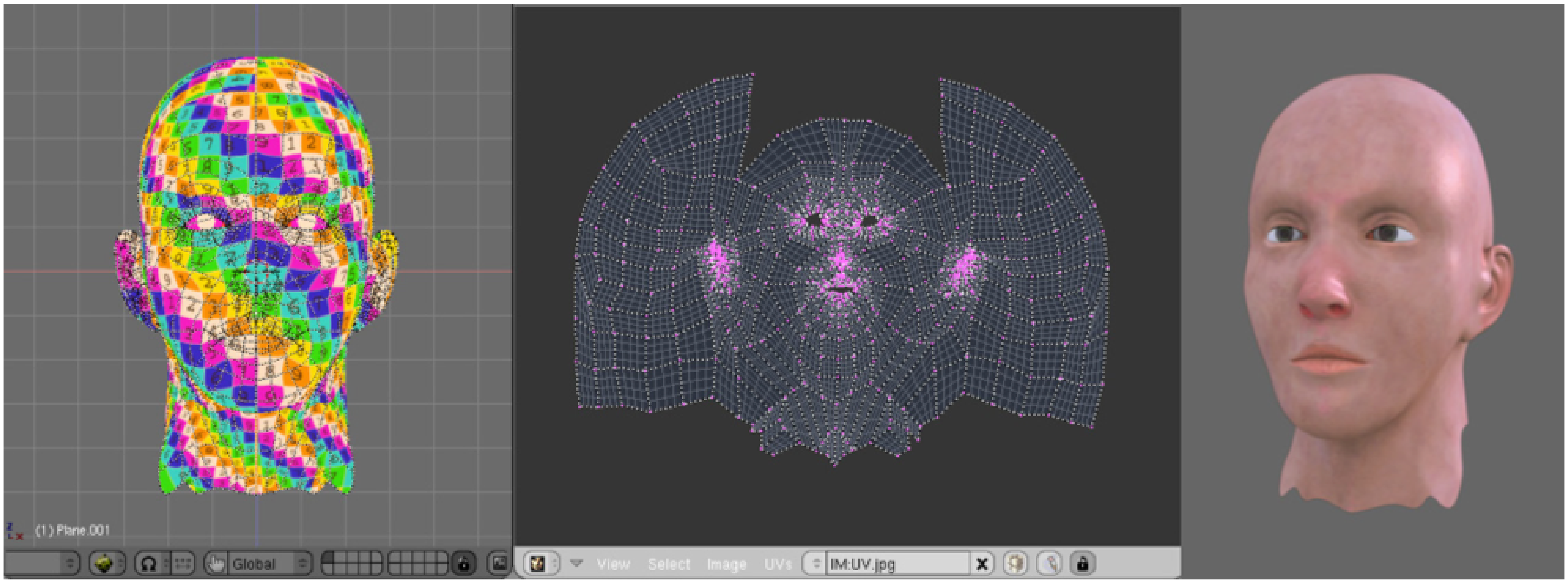
Next Time



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Parameterization

http://cs621.hao-li.com

Thanks!